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The Representation of the Trilinear Kernel in General Orthogonal Polynomials and Some Applications

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Let $\{P_n\}$ $(n \in \mathbb{Z}_+)$ denote the sequence of orthonormal polynomials with respect to the weight $w(x)(-1 \leqslant x \leqslant 1)$. The representation of the kernel

$$\mathscr{D}_n(x, y, z) = \sum_{k=0}^{n} p_k(x)\, p_k(y)\, p_k(z) \qquad (n \in \mathbb{Z}_+;\ -1 \leqslant x, y, z \leqslant 1)$$

is given. We use this result to construct a "double-humpbacked majorant" of the kernel $\mathscr{D}_n(x, y, z)$ to estimate the Lebesgue quasifunction, and to compute the infinite sums

$$\sum_{k=0}^{\infty} \lambda_k\, p_k(x)\, p_k(y)\, p_k(z) \qquad (\lambda_k \in \mathbb{R}^1,\ \lambda_k \neq 0,\ k \in \mathbb{Z}_+),$$

which appear in some problems of mathematical physics and in the theory of group representations.

1. INTRODUCTION

Let $w(x)$ be nonnegative in $[-1, 1]$ and positive almost everywhere on $[-1, 1]$, and suppose

$$\int_{-1}^{1} w(x)\, dx < \infty.$$

We call $w(x)$ a weight function (weight). Associated with $w(x)$ is the sequence of orthonormal polynomials ONSP$\{p_n \equiv p_n(w; x)\}$ $(n \in \mathbb{Z}_+ = \{0, 1, 2, ...\})$, where

$$p_n(x) = k_n x^n + \cdot$$

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has degree n with $k_n = k_n(w) > 0$ and

$$\int_{-1}^{1} p_n(x)\, p_m(x)\, w(x)\, dx = \begin{cases} 1 & \text{for} \quad m = n \\ 0 & \text{for} \quad m \neq n. \end{cases}$$

The orthonormal polynomials satisfy the three-term recurrence relation [12, p. 17; 48, pp. 55–56]

$$xp_n(x) = a_n p_{n+1}(x) + u_n p_n(x) + a_{n-1} p_{n-1}(x) (p_{-1}(x) \equiv 0, n \in \mathbb{Z}_+), \tag{1.1}$$

where, for $n \in \mathbb{Z}_+$, the recurrence coefficients $a_n \equiv a_n(w)$ and $u_n = u_n(w)$ satisfy

$$a_n = \frac{k_n}{k_{n+1}}$$

and

$$u_n = \int_{-1}^{1} xp_n^2(x)\, w(x)\, dx.$$

Note that

$$a_n \leqslant 1, \qquad |u_n| \leqslant 1 \qquad (n \in \mathbb{Z}_+) \tag{1.2}$$

and, moreover (Rahmanov's theorem [46, 28, 47]),

$$\lim_{n \to \infty} a_n = \tfrac{1}{2}, \qquad \lim_{n \to \infty} u_n = 0. \tag{1.3}$$

A system of orthogonal polynomials for which the recurrence coefficients satisfy (1.3) belongs to the class $M \equiv M(1, 0)$, introduced by Nevai [31].

Let us define

$$\mathcal{N}_n = 1 + \sum_{k=0}^{n} |a_k - \tfrac{1}{2}| + \sum_{k=0}^{n} |u_k| \qquad (n \in \mathbb{Z}_+), \tag{1.4}$$

where a_n, u_n $(n \in \mathbb{Z}_+)$ are the coefficients of the recursion formula (1.1). The estimate

$$\mathcal{N}_n = o(n) \qquad (n \to \infty) \tag{1.5}$$

holds by virtue of (1.3).

By Favard's theorem [12, p. 60] the recursion formula (1.1) completely determines the orthonormal sequence $\{p_n\}$ $(n \in \mathbb{Z}_+)$; therefore, many investigations are devoted to ONSP $\{p_n\}$ $(n \in \mathbb{Z}_+)$, defined by the recursion formula. This is of interest in scattering theory, in chain sequences, or in spectral theory of Jacobi matrices [1, 6, 8, 15, 24, 34].

With appropriate conditions of the recurrence coefficients one can obtain properties of the weight w, asymptotics for $\{p_n\}$ and the zeros of $p_n(x)$, weighted estimation, and so on [6, 8, 9, 15, 16, 19, 24, 31–34, 49, 50].

We also remark that the polynomials are generalized eigenfunctions in l^2 for the Sturm–Liouville singular difference operator (see [6, Chap. VII, Sect. 1])

$$\begin{aligned}(Lw)_n &= a_{n-1}w_{n-1} + u_n w_n + a_n w_{n+1} \\ &= (\nabla[a(\Delta w)])_n + q_n w_n \qquad (w_{-1}=0,\, n \in \mathbb{Z}_+),\end{aligned}$$

where

$$\left.\begin{aligned} w = \{w_n\},\ \Delta w_n = w_{n+1} - w_n,\ \nabla w_n = w_n - w_{n-1} \\ a = \{a_n\},\ q_n = a_{n-1} + a_n + u_n \end{aligned}\right\} \qquad (n \in \mathbb{Z}_+).$$

A fundamental role for the treatment of expansions of functions in orthogonal polynomials is played by the Christoffel–Darboux summation formula

$$\begin{aligned}\mathscr{D}_n(x, y) &= \sum_{k=0}^{n} p_k(x)\, p_k(y) \\ &= a_n \frac{p_{n+1}(y)\, p_n(x) - p_n(y)\, p_{n+1}(x)}{y - x} \quad (n \in \mathbb{Z}_+;\, y, x \in [-1, 1]).\end{aligned}$$

We consider the trilinear kernel

$$\mathscr{D}_n(x, y, z) = \sum_{k=0}^{n} p_k(x)\, p_k(y)\, p_k(z) \qquad (n \in \mathbb{Z}_+;\, -1 \leqslant x, y, z \leqslant 1), \tag{1.6}$$

which possesses the reproduction property: for every polynomial

$$\pi_n(x) = \sum_{k=0}^{n} c_k\, p_k(x) \qquad (n \in \mathbb{Z}_+;\, x \in [-1, 1])$$

the relation

$$\begin{aligned}&\int_{-1}^{1} \pi_n(z)\, \mathscr{D}_n(x, y, z)\, w(z)\, dz \\ &\qquad = \sum_{k=0}^{n} c_k\, p_k(x)\, p_k(y) \qquad (n \in \mathbb{Z}_+;\, x, y \in [-1, 1])\end{aligned}$$

holds.

The following problem often arises in mathematical physics and in the theory of group representations [10, 45, 51]: determine the sum of the series

$$\sum_{k=0}^{n} \lambda_k p_k(x) p_k(y) p_k(z) \equiv K(x, y, z; \lambda), \tag{1.7}$$

where

$$\lambda = \left\{ \lambda_n;\, \lambda_0 = \frac{1}{p_0},\, \lambda_n \in \mathbb{R}^1,\, \lambda_n \neq 0,\, n \in \mathbb{Z}_+ \right\} \tag{1.8}$$

is any given sequence. For example, in [10, 45] for the orthogonal ultraspherical polynomials $\{C_n^{\varkappa}(x)\}$ $(\varkappa > 0)$ it is shown that

$$\sum_{k=0}^{n} (k+\varkappa) \left\{ \frac{\Gamma(k+1)}{\Gamma(k+2\varkappa)} \right\}^{1/2} C_k^{\varkappa}(\cos\alpha)\, C_k^{\varkappa}(\cos\beta)\, C_k^{\varkappa}(\cos\gamma)$$

$$= 2^{-2\varkappa} \pi [\Gamma(\varkappa)]^{-4} [\sin\alpha \sin\beta \sin\gamma]^{1-2\varkappa}$$

$$\times \left\{ \sin\frac{\alpha+\beta+\gamma}{2} \sin\frac{\beta+\gamma-\alpha}{2} \sin\frac{\alpha-\beta+\gamma}{2} \sin\frac{\alpha+\beta-\gamma}{2} \right\}^{\varkappa-1},$$

where $0 < \alpha,\ \beta,\ \gamma < \pi$, and a triangle can be drawn with sides α, β, γ, assuming that the sum of any two of the sides is less than or equal to π; otherwise the infinite sum is 0.

The main purpose of this paper is to state a representation of the kernel $\mathscr{D}_n(x, y, z)$. We apply this result to the computation of the infinite trilinear sums (1.7), (1.8) and to the construction of a "double-humpbacked majorant" of the trilinear kernel $\mathscr{D}_n(x, y, z)$.

Some of our results were discussed in [35–37]. Other applications of the representation of the trilinear kernel $\mathscr{D}_n$ and of the estimation of a "double-humbacked majorant" will be given in forthcoming articles.

2. A Formula for the Kernel: Computation of the Trilinear Sum

Let us define the function $T_n(x)$ by the formulae

$$T_n(x) = \cos(n \arccos x)\ (n = 1, 2, \ldots), \qquad T_0(x) = 1\ (-1 \leqslant x \leqslant 1).$$

Then $T_n(x)$ is a polynomial of degree n with a positive leading coefficient. The system

$$p_n^{(0)}(x) = \sqrt{\frac{2}{\pi}}\, T_n(x)\ (n = 1, 2, \ldots),$$

$$p_0^{(0)}(x) = \frac{1}{\sqrt{\pi}}\, T_0(x)\ (-1 \leqslant x \leqslant 1)$$

is the system of orthonormal polynomials with weight

$$w_0(x)=\frac{1}{\sqrt{1-x^2}} \qquad (-1<x<1)$$

(Chebyshev polynomials of the first kind). We have

$$xp_n^{(0)}(x)=\tfrac{1}{2}p_{n+1}^{(0)}(x)+\tfrac{1}{2}p_{n-1}^{(0)}(x) \qquad (n\geqslant 2)$$

and, consequently, the system $\{p_n^{(0)}(x)\}$ $(n\in\mathbb{Z}_+)$ belongs to Nevai's M-class. Putting

$$x=\cos\alpha,\ y=\cos\beta,\ \zeta_+=\cos(\alpha-\beta),\ \zeta_-=\cos(\alpha+\beta),$$

we obtain

$$\sum_{k=0}^{n} T_k(x)\,T_k(y)\,T_k(z)$$
$$=\tfrac{1}{2}\sum_{k=0}^{n} T_k(\zeta_+)\,T_k(z)+\tfrac{1}{2}\sum_{k=0}^{n} T_k(\zeta_-)\,T_k(z)$$

and by the Christoffel–Darboux summation formula the sums on the right-hand side becomes a sum of two fractions with denominators $z-\zeta_-$ and $z-\zeta_+$. So, if the recurrence relation (1.1) belongs to Nevai's M-class, one expects that the trilinear kernel has two peaks near $z=\zeta_-$ and $z=\zeta_+$.

Note that

$$\zeta_-=xy-\sqrt{(1-x^2)(1-y^2)}, \qquad \zeta_+=xy+\sqrt{(1-x^2)(1-y^2)}$$
$$(-1\leqslant x, y\leqslant 1) \tag{2.1}$$

so

$$(\zeta_- -z)(z-\zeta_+)=1+2xyz-x^2-y^2-z^2 \qquad (-1\leqslant x, y, z\leqslant 1) \tag{2.2}$$

and

$$(\zeta_- -z)(z-\zeta_+)=4\sin\frac{\alpha+\beta+\gamma}{2}\sin\frac{\alpha+\beta-\gamma}{2}\sin\frac{\beta+\gamma-\alpha}{2}\sin\frac{\alpha-\beta+\gamma}{2}, \tag{2.3}$$

where $x=\cos\alpha$, $y=\cos\beta$, $z=\cos\gamma$ $(0<\alpha, \beta, \gamma<\pi)$.

The next statement plays a fundamental role throughout this paper.

LEMMA 2.1. *For a general* ONSP $\{p_n\}$ $(n\in\mathbb{Z}_+)$ *the following representation is valid*:

$$(z-\zeta_-)(\zeta_+-z)\,\mathscr{D}_n(x,y,z)$$
$$=A_n(x,y,z)+B_n(x,y,z)+E_n(x,y,z)\ (n\in\mathbb{Z}_+;\ -1\leqslant x,y,z\leqslant 1), \tag{2.4}$$

where

$$\begin{aligned}A_n(x,y,z)=2a_n^3\,&p_{n+1}(x)\,p_{n+1}(y)\,p_{n+1}(z)\\&+[1+2a_{n-1}^3-3(a_{n-1}^2+a_n^2)]\,p_n(x)\,p_n(y)\,p_n(z)\\&+2a_{n-1}a_n^2[p_{n+1}(x)\,p_{n-1}(y)\,p_{n+1}(z)\\&+p_{n+1}(x)\,p_{n+1}(y)\,p_{n-1}(z)+p_{n-1}(x)\,p_{n+1}(y)\,p_{n+1}(z)]\\&-a_na_{n+1}[p_{n+2}(x)\,p_n(y)\,p_n(z)\\&+p_n(x)\,p_{n+2}(y)\,p_n(z)+p_n(x)\,p_n(y)\,p_{n+2}(z)];\end{aligned} \tag{2.5}$$

$$\begin{aligned}B_n(x,y,z)=\sum_{k=0}^{n-2}\,&[(a_k-\tfrac{1}{2})(2a_k^2-2a_k-1)\\&+(a_{k-1}-\tfrac{1}{2})(2a_{k-1}^2-2a_{k-1}-1)]\,p_k(x)\,p_k(y)\,p_k(z)\\&+2\sum_{k=0}^{n-2}a_ka_{k+1}(a_{k+1}-\tfrac{1}{2})[p_{k+2}(x)\,p_k(y)\,p_{k+2}(z)\\&+p_{k+2}(x)\,p_{k+2}(y)\,p_k(z)+p_k(x)\,p_{k+2}(y)\,p_{k+2}(z)]\\&+2\sum_{k=0}^{n-1}a_ka_{k+1}(a_k-\tfrac{1}{2})[p_{k+2}(x)\,p_k(y)\,p_k(z)\\&+p_k(x)\,p_{k+2}(y)\,p_k(z)+p_k(x)\,p_k(y)\,p_{k+2}(z)];\end{aligned} \tag{2.6}$$

and

$$E_n(x,y,z)=\sum_{k=0}^{n}e_k(x,y,z)\qquad(n\in\mathbb{Z}_+;\ x,y,z\in[-1,1]) \tag{2.7}$$

with

$$\begin{aligned}e_k(x,y,z)=2u_k\{&[a_k\,p_{k+1}(x)+a_{k-1}\,p_{k-1}(x)]\\&\times[a_k\,p_{k+1}(y)+a_{k-1}\,p_{k-1}(y)]\,p_k(z)\\&+[a_k\,p_{k+1}(x)+a_{k-1}\,p_{k-1}(x)]\,p_k(y)\\&\times[a_k\,p_{k+1}(z)+u_k\,p_k(z)+a_{k-1}\,p_{k-1}(z)]\\&-\tfrac{3}{2}u_k\,p_k(x)\,p_k(y)\,p_k(z)\end{aligned}$$

$$+ p_k(x)[a_k p_{k+1}(y) + u_k p_k(y) + a_{k-1} p_{k-1}(y)]$$
$$\times [a_k p_k(z) + u_k p_k(z) + a_{k-1} p_{-1}(z)]\}$$
$$- a_k(u_{k-1} + u_k)[p_{k+1}(x) p_k(y) p_k(z)$$
$$+ p_k(x) p_{k+1}(y) p_k(z) + p_k(x) p_k(y) p_{k+1}(z)]$$
$$- a_{k-1}(u_{k-1} + u_k)[p_{k-2}(x) p_k(y) p_k(z)$$
$$+ p_k(x) p_{k-2}(y) p_k(z) + p_k(x) p_k(y) p_{k-2}(z)]. \tag{2.8}$$

Here $\{a_k\}$, $\{u_k\}$ $(k \in \mathbb{Z}_+)$ *are recurrence coefficients in* (1.1) *for the polynomials* $p_n(x)$ $(n \in \mathbb{Z}_+)$ *and* $-1 \leqslant x, y, z \leqslant 1$.

Proof. In view of the three-term recurrence relation (1.1) it is not difficult to see that

$$x^2 p_n(x) = a_n a_{n+1} p_{n+2}(x) + a_n(u_n + u_{n+1}) p_{n+1}(x)$$
$$+ (a_{n-1}^2 + a_n^2 + u_n^2) p_n(x) + a_{n-1}(u_{n-1} + u_n) p_{n-1}(x)$$
$$+ a_{n-2} a_{n-1} p_{n-2}(x) \qquad (n \in \mathbb{Z}_+;\, p_{-1}(x) \equiv 0, p_{-2}(x) \equiv 0).$$

By (2.2) we have

$$(z - \zeta_-)(\zeta_+ - z) p_k(x) p_k(y) p_k(z)$$
$$= (1 + 2xyz - x^2 - y^2 - z^2) p_k(x) p_k(y) p_k(z)$$
$$= p_k(x) p_k(y) p_k(z) + 2a_k^3 p_{k+1}(x) p_{k+1}(y) p_{k+1}(z)$$
$$+ 2a_{k-1} a_k^2 p_{k+1}(x) p_{k-1}(y) p_{k+1}(z)$$
$$+ 2a_{k-1} a_k^2 p_{k+1}(x) p_{k+1}(y) p_{k-1}(z)$$
$$+ 2a_{k-1}^2 a_k p_{k+1}(x) p_{k-1}(y) p_{k-1}(z)$$
$$+ 2a_{k-1} a_k^2 p_{k-1}(x) p_{k+1}(y) p_{k+1}(z)$$
$$+ 2a_k a_{k-1}^2 p_{k-1}(x) p_{k-1}(y) p_{k+1}(z)$$
$$+ 2a_{k-1}^2 a_k p_{k-1}(x) p_{k+1}(y) p_{k-1}(z)$$
$$+ 2a_{k-1}^3 p_{k-1}(x) p_{k-1}(y) p_{k-1}(z) + e_k(x, y, z)$$
$$- a_k a_{k+1} p_{k+2}(x) p_k(y) p_k(z) - (a_{k-1}^2 + a_k^2) p_k(x) p_k(y) p_k(z)$$
$$- a_{k-2} a_{k-1} p_{k-2}(x) p_k(y) p_k(z) - a_k a_{k+1} p_k(x) p_{k+2}(y) p_k(z)$$
$$- (a_{k-1}^2 + a_k^2) p_k(x) p_k(y) p_k(z) - a_{k-2} a_{k-1} p_k(x) p_{k-2}(y) p_k(z)$$
$$- a_k a_{k+1} p_k(x) p_k(y) p_{k+2}(z) - (a_{k-1}^2 + a_k^2) p_k(x) p_k(y) p_k(z)$$
$$- a_{k-2} a_{k-1} p_k(x) p_k(y) p_{k-2}(z),$$

where $e_k(x, y, z)$ is defined by (2.8). We regroup similar terms of the relation

$$\begin{aligned}
&(z-\zeta_-)(\zeta_+-z)\, p_k(x)\, p_k(y)\, p_k(z) \\
&\quad = e_k(x, y, z) + 2a_k^3\, p_{k+1}(x)\, p_{k+1}(y)\, p_{k+1}(z) \\
&\qquad + 2a_{k-1}^3\, p_{k-1}(x)\, p_{k-1}(y)\, p_{k-1}(z) \\
&\qquad + [1-3(a_{k-1}^2+a_k^2)]\, p_k(x)\, p_k(y)\, p_k(z) \\
&\qquad + 2a_{k-1}a_k^2\, p_{k+1}(x)\, p_{k-1}(y)\, p_{k+1}(z) - a_{k-2}a_{k-1}\, p_k(x)\, p_{k-2}(y)\, p_k(z) \\
&\qquad + 2a_{k-1}a_k^2\, p_{k+1}(x)\, p_{k+1}(y)\, p_{k-1}(z) - a_{k-2}a_{k-1}\, p_k(x)\, p_k(y)\, p_{k-2}(z) \\
&\qquad + 2a_{k-1}a_k^2\, p_{k-1}(x)\, p_{k+1}(y)\, p_{k+1}(z) - a_{k-2}a_{k-1}\, p_{k-2}(x)\, p_k(y)\, p_k(z) \\
&\qquad + 2a_{k-1}^2a_k\, p_{k+1}(x)\, p_{k-1}(y)\, p_{k-1}(z) - a_ka_{k+1}\, p_{k+2}(x)\, p_k(y)\, p_k(z) \\
&\qquad + 2a_{k-1}^2a_k\, p_{k-1}(x)\, p_{k-1}(y)\, p_{k+1}(z) - a_ka_{k+1}\, p_k(x)\, p_k(y)\, p_{k+2}(z) \\
&\qquad + 2a_{k-1}^2a_k\, p_{k-1}(x)\, p_{k+1}(y)\, p_{k-1}(z) - a_ka_{k+1}\, p_k(x)\, p_{k+2}(y)\, p_k(z).
\end{aligned}$$

For the proof of formulae (2.4)–(2.8) we consider the following "basic" terms (the others are treated in a similar manner):

$$\begin{aligned}
\Sigma_1 = \sum_{k=0}^{n} \{&2a_k^3\, p_{k+1}(x)\, p_{k+1}(y)\, p_{k+1}(z) \\
&+ [1-3(a_{k-1}^2+a_k^2)]\, p_k(x)\, p_k(y)\, p_k(z) \\
&+ 2a_{k-1}^3\, p_{k-1}(x)\, p_{k-1}(y)\, p_{k-1}(z)\},
\end{aligned}$$

$$\begin{aligned}
\Sigma_2 = \sum_{k=0}^{n} [&2a_{k-1}a_k^2\, p_{k+1}(x)\, p_{k-1}(y)\, p_{k+1}(z) \\
&- a_{k-2}a_{k-1}\, p_k(x)\, p_{k-2}(y)\, p_k(z)],
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_3 = \sum_{k=0}^{n} [&2a_{k-1}^2a_k\, p_{k+1}(x)\, p_{k-1}(y)\, p_{k-1}(z) \\
&- a_ka_{k+1}\, p_{k+2}(x)\, p_k(y)\, p_k(z)].
\end{aligned}$$

Using $p_{-1}(x)=0$ and $2a_k^3-3a_k^2+\frac{1}{2}=(a_k-\frac{1}{2})(2a_k^2-2a_k-1)$ $(k\in\mathbb{Z}_+)$, we have

$$\sum\nolimits_1 = 2a_n^3 p_{n+1}(x)\, p_{n+1}(y)\, p_{n+1}(z) + [2a_{n-1}^3 + 1 - 3(a_{n-1}^2 + a_n^2)]\, p_n(x)\, p_n(y)\, p_n(z) + \sum_{k=0}^{n-2} [(a_k - \tfrac{1}{2})(2a_k^2 - 2a_k - 1) + (a_{k-1} - \tfrac{1}{2})(2a_k^2 - 1 - 2a_{k-1} - 1)]\, p_k(x)\, p_k(y)\, p_k(z).$$

Next, note that $p_{-2}(x) \equiv p_{-1}(x) \equiv 0$; then

$$\sum\nolimits_2 = 2\sum_{k=0}^{n-1} a_k a_{k+1}^2\, p_{k+2}(x)\, p_k(y)\, p_{k+2}(z) - \sum_{k=0}^{n-2} a_k a_{k+1}\, p_{k+2}(x)\, p_k(y)\, p_{k+2}(z)$$
$$= 2a_{n-1} a_n^2\, p_{n+1}(x)\, p_{n-1}(y)\, p_{n+1}(z) + 2\sum_{k=0}^{n-2} a_k a_{k+1}(a_{k+1} - \tfrac{1}{2})\, p_{k+2}(x)\, p_k(y)\, p_{k+2}(z).$$

In a similar way

$$\sum\nolimits_3 = -a_n a_{n+1}\, p_{n+2}(x)\, p_n(y)\, p_n(z) + 2\sum_{k=0}^{n-1} a_k a_{k+1}(a_k - \tfrac{1}{2})\, p_{k+2}(x)\, p_k(y)\, p_k(z).$$

Formulae (2.4)–(2.8) are a consequence of the last three formulae.

Remarks. (1) The above result holds in particular for D.P. in Nevai's class.

(2) For an even weight $w(x)$ on $[-1, 1]$ formulae (2.4)–(2.8) were announced in [35].

(3) A similar result can be obtained for the trilinear form

$$\sum_{k=0}^{n} q_k(x)\, q_k(y)\, q_k(z) \qquad (n \in \mathbb{Z}_+;\ -1 \leqslant x, y, z \leqslant 1),$$

where $\{q_n\}$ $(n \in \mathbb{Z}_+)$ is the system of the functions or the polynomials of the second kind [45, 51].

(4) Many papers are devoted to the investigations and the applications of the kernel $\mathscr{D}_n(x, y, z)$ for the classical polynomials and its generalizations (cf. [4, 7, 10, 11, 13, 14, 17, 18, 20–23, 25–27, 38–44, 51]).

(5) The following curious result can be inferred from Lemma 2.1.

COROLLARY 2.2. *For an ONSP $\{p_n\}$ $(n \in \mathbb{Z}_+)$ with respect to an even weight function $w(x)$ on $[-1, 1]$, we have the representation for all $x \in [-1, 1]$ and $n \in \mathbb{Z}_+$:*

$$\begin{aligned}
2(x-1)^2 \left(x+\tfrac{1}{2}\right) \sum_{k=0}^{n} p_k^3(x) \\
&= 2a_n^3 p_{n+1}^3(x) + [1 + 2a_{n-1}^3 - 3(a_{n-1}^2 + a_n^2)]\, p_n^3(x) \\
&\quad + 6a_{n-1} a_n^2 p_{n+1}^2(x)\, p_{n-1}(x) - 3a_n a_{n+1} p_{n+2}(x)\, p_n^2(x) \\
&\quad + 6 \sum_{k=0}^{n-2} a_k a_{k+1} \left(a_{k+1} - \tfrac{1}{2}\right) p_k(x)\, p_{k+2}^2(x) \\
&\quad + \sum_{k=0}^{n-2} \left[\left(a_k - \tfrac{1}{2}\right)(2a_k^2 - 2a_k - 1)\right. \\
&\quad \left. + \left(a_{k-1} - \tfrac{1}{2}\right)(2a_{k-1}^2 - 2a_{k-1} - 1)\right] p_k^3(x) \\
&\quad + 6 \sum_{k=0}^{n-1} a_k a_{k+1} \left(a_k - \tfrac{1}{2}\right) p_k^2(x)\, p_{k+2}(x).
\end{aligned}$$

In fact, since $w(x)$ is even $u_n = 0$ $(n \in \mathbb{Z}_+)$ and for $x = y = z$

$$(z - \zeta_-)(\zeta_+ - z) = (x-1)^2 (2x+1) \qquad (-1 \leqslant x \leqslant 1).$$

Throughout this paper we consider ONSP $\{p_n\}$ $(n \in \mathbb{Z}_+)$ with respect to the weight $w(x)$ on $[-1, 1]$, for which the following hypothesis is valid: there exists a positive L_w^1-integrable function $\varphi(x)$ such that

$$|p_n(x)| \leqslant \varphi(x) \qquad (n \in \mathbb{Z}_+;\ -1 < x < 1). \tag{2.9}$$

The following statement can be inferred from Lemma 2.1

LEMMA 2.3. *If (1.3) and (2.9) hold, then*

$$\begin{aligned}
&|(z - \zeta_-)(\zeta_+ - z)\, \mathscr{D}_n(x, y, z)| \\
&\quad \leqslant |A_n(x, y, z) + B_n(x, y, z) + E_n(x, y, z)| \\
&\quad \leqslant C\mathscr{N}_n \varphi(x)\, \varphi(y)\, \varphi(z) \qquad (n \in \mathbb{Z}_+;\ -1 < x, y, z < 1).
\end{aligned} \tag{2.10}$$

We indicate some examples of ONSP $\{p_n\}$ $(n \in \mathbb{Z}_+)$, satisfying the condition (2.9). Throughout C, C_α, $C_{\alpha\beta}$, ... denote positive constants, independent of $n \in \mathbb{Z}_+$ and $x, y, z \in (-1, 1)$. The same symbol does not necessarily denote the same constant from line to line.

1. Let $w_{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$ be a Jacobi weight on $[-1, 1]$ with

parameters α and $\beta(\alpha, \beta > -1)$. For classical orthonormal (with weight $w_{\alpha,\beta}$, Jacobi polynomials: $\{p_n^{(\alpha,\beta)}(x)\}$ $(n \in \mathbb{Z}_+)$

$$|p_n^{(\alpha,\beta)}(x)| \leqslant C_{\alpha\beta}(1-x)^{-(\alpha/2+1/4)}\,(1+x)^{-(\beta/2+1/4)}$$
$$(n \in \mathbb{Z}_+;\ -1 < x < 1;\ \alpha, \beta \geqslant -\tfrac{1}{2})$$

and

$$|p_n^{\alpha,\beta}(x)| \leqslant C_{\alpha\beta} \qquad (n \in \mathbb{Z}_+;\ -1 < x < 1;\ -1 < \alpha, \beta < -\tfrac{1}{2})$$

are valid. In this case for the recurrence coefficients

$$a_n^2 = [a_n^{(\alpha,\beta)}]^2 = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta+2)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta+2)^2\,(2n+\alpha+\beta+1)}$$
$$= \frac{1}{4} + \frac{1-2(\alpha^2+\beta^2)}{16n^2} + O\left(\frac{1}{n^3}\right),$$
$$u_n = u_n^{(\alpha,\beta)} = \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)} = \frac{\beta^2-\alpha^2}{4n^2} + O\left(\frac{1}{n^3}\right)$$

the condition

$$\mathcal{N}_n \equiv \mathcal{N}_n^{(\alpha,\beta)} = 1 + \sum_{k=0}^{n} |a_k^{(\alpha,\beta)} - \tfrac{1}{2}| + \sum_{k=0}^{n} |u_n^{(\alpha,\beta)}| \leqslant C \qquad (n \in \mathbb{Z}_+)$$

is fulfilled.

2. Consider orthonormal system $\{p_n\}$ $(n \in \mathbb{Z}_+)$ generated by the recurrence relation (1.1), where

$$\sum_{k=0}^{n} (k+1)(|1-4a_{k+1}^2| + 2\,|u_k|) \leqslant C \log(n+1) \qquad (n = 1, 2, ...).$$

In [33] (cf. also [50]) P. Nevai proved that there exist positive constants C_1, C_2 independent of $x \in [-1, 1]$ and $n \in \mathbb{Z}_+$ such that

$$|p_n(x)| \leqslant c_1(1-x^2)^{-c_2} \qquad (n \in \mathbb{Z}_+;\ -1 < x < 1).$$

If we suppose

$$\sum_{k=0}^{\infty} (k+1)\,|a_k - \tfrac{1}{2}| + \sum_{k=0}^{\infty} (k+1)\,|u_k| < \infty, \tag{2.11}$$

then

$$|p_n(x)| \leqslant C(1-x^2)^{-1/4}\,[w(x)]^{-1/2} \qquad (n \in \mathbb{Z}_+;\ -1 < x < 1)$$

and

$$\max_{-1 \leqslant x \leqslant 1} |p_n(x)| \leqslant C(n+1) \qquad (n \in \mathbb{Z}_+)$$

hold [15, 31, 50].

In particular, Bernstein–Szegö polynomials [48, pp. 44–45], Askey–Wilson q-polynomials $\{p_n(x; a, b, c, d \mid q)\}$ $\max(|q|, |a|, |b|, |c|, |d| < 1)$ [2, 3, 5], and the orthogonal polynomial system $\{\theta_n^{(a)}(x; q)\}$ $(a > 0, |q| < 1)$ introduced by M. E. H. Ismail and F. S. Mulla [19] satisfy (2.11).

3. In [32], P. Nevai studied ONSP $\{p_n\}$ $(n \in \mathbb{Z}_+)$, defined by the three-term recurrence relation (1.1) with

$$a_n = \frac{1}{2} + \frac{(-1)^n E}{n} + O\left(\frac{1}{n^2}\right), \qquad u_n = \frac{(-1)^n D}{n} + O\left(\frac{1}{n^2}\right),$$

where E, D are absolute constants. In this case there exist three positive numbers a, b, and c such that

$$p_n^2(x) \leqslant c(1-x^2)^{-b} |x|^{-a} \qquad (n \in \mathbb{Z}_+; -1 < x < 1)$$

and

$$w(x) \geqslant c^{-1} |x|^a (1-x^2)^b.$$

Note that for such parameters

$$\mathcal{N}_n \leqslant c \log(n+2) \qquad (n \in \mathbb{Z}_+).$$

If

$$w(x) = |x|^a (1-x^2)^b,$$

then one has a generalized Jacobi polynomial.

Let us define the auxiliary functions

$$\left.\begin{aligned} \widetilde{\mathscr{D}}_n(x, y, z) &= \frac{1}{\varphi(x)\,\varphi(y)\,\varphi(z)}\, \mathscr{D}_n(x, y, z) \\ \widetilde{A}_n(x, y, z) &= \frac{1}{\varphi(x)\,\varphi(y)\,\varphi(z)}\, A_n(x, y, z) \\ \widetilde{B}_n(x, y, z) &= \frac{1}{\varphi(x)\,\varphi(y)\,\varphi(z)}\, B_n(x, y, z) \\ \widetilde{E}_n(x, y, z) &= \frac{1}{\varphi(x)\,\varphi(y)\,\varphi(z)}\, E_n(x, y, z) \\ \widetilde{K}_n(x, y, z; \lambda) &= \frac{1}{\varphi(x)\,\varphi(y)\,\varphi(z)}\, K_n(x, y, z; \lambda) \end{aligned}\right\} \quad (n \in \mathbb{Z}_+; -1 < x, y, z < 1), \tag{2.12}$$

where $\varphi(x)$ is the majorant of $\{p_n\}$ $(n \in \mathbb{Z}_+)$ (cf. (2.9)) and

$$K_n(x, y, z; \lambda) = \sum_{k=0}^{n} \lambda_k p_k(x) p_k(y) p_k(z) \ (n \in \mathbb{Z}_+; -1 < x, y, z < 1) \tag{2.13}$$

are the first partial sums of the orthogonal series (1.7).

THEOREM 2.4. *Let* ONSP $\{p_n\}$ $(n \in \mathbb{Z}_+)$ *satisfy* (1.3), (2.9) *and suppose that*

$$\sum_{k=0}^{\infty} \mathcal{N}_k |\Delta(\lambda_k)| < \infty \tag{2.14}$$

and

$$\lim_{n \to \infty} \lambda_n \mathcal{N}_n = 0 \tag{2.15}$$

hold. Then the kernel

$$\tilde{K}(x, y, z; \lambda) = \lim_{n \to \infty} \tilde{K}_n(x, y, z; \lambda)$$

exists for every $x, y, z \in (-1, 1)$ *and*

$$4 \sin\frac{\alpha+\beta+\gamma}{2} \sin\frac{\alpha+\beta-\gamma}{2} \sin\frac{\beta+\gamma-\alpha}{2} \sin\frac{\alpha-\beta+\gamma}{2} \tilde{K}(\cos\alpha, \cos\beta, \cos\gamma; \lambda)$$
$$= \sum_{k=0}^{\infty} \Delta(\lambda_k)[\tilde{A}_k(\cos\alpha, \cos\beta, \cos\gamma)$$
$$+ \tilde{B}_k(\cos\alpha, \cos\beta, \cos\gamma) + \tilde{E}_k(\cos\alpha, \cos\beta, \cos\gamma)], \tag{2.16}$$

where the series on the right-hand side are absolutely convergent for $0 < \alpha, \beta, \gamma < \pi$.

Proof. By the aid of Abel's summation by parts it can be inferred from (2.13) that

$$K_n(x, y, z; \lambda) = \lambda_n D_n(x, y, z)$$
$$- \sum_{k=0}^{n-1} \Delta(\lambda_k) \mathcal{D}_k(x, y, z) \qquad (-1 \leqslant x, y, z \leqslant 1; n \in \mathbb{Z}_+),$$

where the kernel $\mathcal{D}_n(x, y, z)$ is defined by (1.6). In consequence of the formula (2.4) and the definition (2.12)

$$
\begin{aligned}
(z-\zeta_-)(\zeta_+-z)\,\tilde{K}(x,y,z;\lambda)
&= \lim_{n\to\infty} \lambda_n[\tilde{A}_n(x,y,z)+\tilde{B}_n(x,y,z)+\tilde{E}_n(x,y,z)] \\
&\quad - \lim_{n\to\infty} \sum_{k=0}^{n-1} \Delta(\lambda_k)[\tilde{A}_k(x,y,z)+\tilde{B}_k(x,y,z)+\tilde{E}_k(x,y,z)].
\end{aligned}
$$

In view of the relations (1.3), (2.9), (2.10), (2.15), the first term of the last sum vanishes for all $x, y, z \in (-1, 1)$. The second term can be estimated with

$$C \sum_{k=0}^{\infty} \mathcal{N}_k\, |\Delta(\lambda_k)|,$$

and from (2.14) we obtain that the series

$$\sum_{k=0}^{\infty} \Delta(\lambda_k)[\tilde{A}_k(x,y,z)+\tilde{B}_k(x,y,z)+\tilde{E}_k(x,y,z)]$$

converge absolutely for $x, y, z \in (-1, 1)$. Putting $x = \cos\alpha$, $y = \cos\beta$, $z = \cos\gamma$ and by (2.3), we have the formula (2.16), in accordance with our statement.

3. A Construction of the "Double-Humpbacked Majorant": Estimation of the Lebesgue Quasifunctions

The estimations of Lebesgue functions of the kernel play an important role in the treatment of expansions of functions in orthogonal polynomials. We begin with the construction of the majorant. It is well known [29; 30, p. 262] that the nonnegative function

$$F_n^*(\xi, \eta) \qquad (n \in \mathbb{Z}_+;\ \xi, \eta \in (a, b) \subset (-1, 1))$$

is called a "humpbacked majorant" for the sequence $F_n(\xi, \eta)$ in the variable η at the point ξ if the following conditions are satisfied:

(1) for all $n \in \mathbb{Z}_+$ and $\xi, \eta \in (a, b)$

$$|F_n(\xi, \eta)| \leqslant F_n^*(\xi, \eta);$$

(2) for fixed $n \in \mathbb{Z}_+$, $\xi \in (a, b)$, the function $F_n^*(\xi, \eta)$ is nondecreasing on (a, ξ) and nonincreasing on (ξ, b).

We say that the function $\tilde{\mathcal{D}}_n(x, y, z)$ $(-1 < x, y, z < 1;\ n \in \mathbb{Z}_+)$ (cf. (1.6), (2.12)) has on $(-1, 1)$ a "double-humpbacked majorant" $\tilde{\mathcal{D}}_n^*(x, y, z)$ in the variable z at the points $\zeta_\pm$ (cf. (2.1)) if on each of the intervals $(-1, xy)$ and $(xy, 1)$ it possesses a "humpbacked majorant" at the points ζ_-, ζ_+, respectively; i.e., for fixed x and y, $\tilde{\mathscr{D}}_n^*(x, y, z)$ is nondecreasing on $(-1, \zeta_-)$, nonincreasing on (ζ_-, xy), nondecreasing on (xy, ζ_+), and nonincreasing on $(\zeta_+, 1)$.

The following assertions play a significant role in our estimations.

LEMMA 3.1. *Let* ONSP $\{p_n\}$ $(n \in \mathbb{Z}_+)$ *satisfy the condition* (1.3) *and* (2.9). *Then the function* $\tilde{\mathscr{D}}_n(x, y, z)$ *has on* $(-1, 1)$ *in the variable* z *a "double-humpbacked majorant"* $\tilde{\mathscr{D}}_n^*(x, y, z)$ *at the points* ζ_-, ζ_+ *and, furthermore, the estimation*

$$\int_{-1}^{1} \tilde{\mathscr{D}}_n^*(x, y, z)\, dz \leqslant C \frac{1}{\sqrt{(1-x^2)(1-y^2)}} \mathscr{N}_n \ln \frac{n+1}{\mathscr{N}_n}$$

$$(n \in \mathbb{Z}_+;\ -1 < x, y < 1) \tag{3.1}$$

holds, where the constant $C > 0$ *is independent of* $n \in \mathbb{Z}_+$ *and* $x, y \in (-1, 1)$.

Proof. We show, that there exist "humpbacked majorants" for the function $\tilde{\mathscr{D}}_n(x, y, z)$ on $(-1, xy)$ and $(xy, 1)$ at the points ζ_- and ζ_+, respectively. We construct the "humpbacked majorant" for $\tilde{\mathscr{D}}_n(x, y, z)$ on $(xy, 1)$ at the point ζ_+; on the interval $(-1, xy)$ the construction can be deduced in a similar way.

At first, consider the case

$$\zeta_+ = xy + \sqrt{(1-x^2)(1-y^2)} \neq 1, \qquad \text{i.e., } x \neq y.$$

Put

$$\delta_n = \frac{\mathscr{N}_n}{n+1} \sqrt{(1-x^2)(1-y^2)}, \qquad \mathscr{E}_n = \frac{\mathscr{N}_n}{n+1} (1 - \zeta_+). \tag{3.2}$$

By the relation (1.5) $\delta_n \to 0$, $\mathscr{E}_n \to 0$ $(n \to \infty)$. It follows from (1.3), (1.6), (2.9), (2.10) that the following estimations are valid:

$$|\tilde{\mathscr{D}}_n(x, y, z)| \leqslant \begin{cases} C(n+1) & \text{for all } x, y, z \in (-1, 1) \quad (3.3) \\ C \dfrac{\mathscr{N}_n}{|(z-\zeta_-)(\zeta_+ - z)|} & \text{for all } x, y, z \in (-1, 1), \text{ satisfying the condition } |(z-\zeta_-)(\zeta_+ - z)| > 0, \quad (3.4) \end{cases}$$

where the constants $C>0$ do not depend on $n\in\mathbb{Z}_+$ and $x, y, z\in(-1,1)$. Thus we define the "humpbacked majorant" by

$$\tilde{\mathscr{D}}_n^*(x,y,z)=\begin{cases} C\mathcal{N}_n\dfrac{1}{\sqrt{(1-x^2)(1-y^2)}}\dfrac{1}{\zeta_+-z}, & \text{if } xy\leqslant z<\zeta_+-\delta_n \\ C(n+1)\dfrac{1}{(1-x^2)(1-y^2)}, & \text{if } \zeta_+-\delta_n\leqslant z<\zeta_+ \\ C(n+1)\dfrac{1}{\sqrt{(1-x^2)(1-y^2)}}\dfrac{1}{1-\zeta_+}, & \text{if } \zeta_+\leqslant z<\zeta_++\mathscr{E}_n \\ C\mathcal{N}_n\dfrac{1}{\sqrt{(1-x^2)(1-y^2)}}\dfrac{1}{z-\zeta_+}, & \text{if } \zeta_++\mathscr{E}_n\leqslant z<1, \end{cases}$$

where the constants $C>0$ are independent of $n\in\mathbb{Z}_+$ and $x, y, z\in(-1,1)$. In fact, when $xy\leqslant z<1$, then

$$|(z-\zeta_-)(\zeta_+-z)|\geqslant\sqrt{(1-x^2)(1-y^2)}\;|z-\zeta_+|.$$

So, for all $x, y\in(-1,1)$ and $n\in\mathbb{Z}_+$ the estimate

$$|\tilde{\mathscr{D}}_n(x,y,z)|\leqslant\tilde{\mathscr{D}}_n^*(x,y,z)\qquad(n\in\mathbb{Z}_+;\,xy\leqslant z<1)$$

holds. Next, by the defining relation

$$\frac{\partial\tilde{\mathscr{D}}_n^*(x,y,z)}{\partial z}>0\ (xy\leqslant z<\zeta_+-\delta_n),\qquad\frac{\partial\tilde{\mathscr{D}}_n^*(x,y,z)}{\partial z}<0\ (\zeta_++\mathscr{E}_n\leqslant z<1)$$

and

$$\tilde{\mathscr{D}}_n^*(x,y,\zeta_+-\delta_n)=C\frac{n+1}{(1-x^2)(1-y^2)},$$

$$\tilde{\mathscr{D}}_n^*(x,y,\zeta_++\mathscr{E}_n)=C(n+1)\frac{1}{\sqrt{(1-x^2)(1-y^2)}}\frac{1}{1-\zeta_+}.$$

Consequently, the function $\tilde{\mathscr{D}}_n^*(x,y,z)$ is nondecreasing on (xy,ζ_+) and nonincreasing on $(\zeta_+,1)$. So $\tilde{\mathscr{D}}_n^*(x,y,z)$ is a "humpbacked majorant" for $\tilde{\mathscr{D}}_n(x,y,z)$ on $(xy,1)$ at the point ζ_+.

Now consider

$$I_n(x,y)=\int_{xy}^{1}\tilde{\mathscr{D}}_n^*(x,y,z)\,dz\qquad(n\in\mathbb{Z}_+;\,-1<x,y<1).$$

This integral can be estimated in the following way; in virtue of the majorant $\tilde{\mathscr{D}}_n^*(x, y, z)$

$$I_n(x, y) \leqslant C\mathcal{N}_n \frac{1}{\sqrt{(1-x^2)(1-y^2)}} \int_{xy}^{\zeta_+ - \delta_n} \frac{dz}{\zeta_+ - z}$$
$$+ C(n+1) \frac{1}{\sqrt{(1-x^2)(1-y^2)}} \int_{\zeta_+ - \delta_n}^{\zeta_+} dz$$
$$+ C(n+1) \frac{1}{\sqrt{(1-x^2)(1-y^2)}} \frac{1}{1-\zeta_+} \int_{\zeta_+}^{\zeta_+ + \mathscr{E}_n} dz$$
$$+ C\mathcal{N}_n \frac{1}{\sqrt{(1-x^2)(1-y^2)}} \int_{\zeta_+ + \mathscr{E}_n}^{1} \frac{dz}{z - \zeta_+}.$$

Hence, using (3.2), we have

$$I_n(x, y) \leqslant C\mathcal{N}_n \frac{1}{\sqrt{(1-x^2)(1-y^2)}} \ln \frac{\sqrt{(1-x^2)(1-y^2)}}{\delta_n}$$
$$+ C(n+1) \frac{1}{(1-x^2)(1-y^2)} \delta_n$$
$$+ C(n+1) \frac{1}{\sqrt{(1-x^2)(1-y^2)}} \frac{1}{1-\zeta_+} \mathscr{E}_n$$
$$+ C\mathcal{N}_n \frac{1}{\sqrt{(1-x^2)(1-y^2)}} \ln \frac{1-\zeta_+}{\mathscr{E}_n}$$
$$\leqslant C \frac{1}{\sqrt{(1-x^2)(1-y^2)}} \mathcal{N}_n \ln \frac{n+1}{\mathcal{N}_n},$$

where the constant $C > 0$ is independent of $n \in \mathbb{Z}_+$ and $x, y \in (-1, 1)$, in accordance with out statement.

We still must consider the remaining case

$$\zeta_+ = 1, \qquad \text{i.e., } x = y, \zeta_- = 2x^2 - 1.$$

In this case we define the "humpbacked majorant" by

$$\tilde{\mathscr{D}}_n^*(x, y, z) = \begin{cases} C\mathcal{N}_n \dfrac{1}{1-x^2} \dfrac{1}{1-z}, & \text{if } \; x^2 \leqslant z < 1 - \delta_n \\ C(n+1) \dfrac{1}{(1-x^2)^2}, & \text{if } \; 1 - \delta_n \leqslant z < 1, \end{cases}$$

where $C > 0$ are the absolute constants and $\delta_n = (\mathcal{N}_n/(n+1))(1-x^2)$.

Obviously, the function $\tilde{\mathscr{D}}_n(x, x, z)$ is nondecreasing on $(x^2, 1-\delta_n)$ and nonincreasing on $(1-\delta_{n,1})$. Furthermore, as above, we have by straightforward calculation

$$\int_{x^2}^{1} \mathscr{D}_n^*(x, x, z)\, dz \leqslant C\mathscr{N}_n \frac{1}{1-x^2} \int_{x^2}^{1-\delta_n} \frac{dz}{1-z} + C(n+1) \frac{1}{(1-x^2)^2} \int_{1-\delta_n}^{1} dz$$
$$\leqslant C\mathscr{N}_n \frac{1}{1-x^2} \ln \frac{1-x^2}{\delta_n} + C(n+1) \frac{1}{(1-x^2)^2} \delta_n.$$

By the defining relation

$$\int_{x^2}^{1} \tilde{\mathscr{D}}_n^*(x, x, z)\, dz \leqslant C\mathscr{N}_n \frac{1}{1-x^2} \ln \frac{n+1}{\mathscr{N}_n},$$

where the constant $C>0$ is independent of $n \in \mathbb{Z}_+$ and $x \in (-1, 1)$. It coincides with (3.1) as $x=y$.

We have completed the proof of our assertion.

By virtue, of "symmetry" of the function $\tilde{\mathscr{D}}_n^*(x, y, z)$ we can construct "double-humpbacked majorants" in the variables x and y.

COROLLARY 3.2. *Assume that* ONSP $\{p_n\}$ $(n \in \mathbb{Z}_+)$ *with the weight* $w(x)$ *satisfy* (1.3) *and* (2.9). *Then for Lebesgue's quasifunctions* $\tilde{L}_n$ *the following estimations hold*:

$$\tilde{L}_n(x, y) = \int_{-1}^{1} |\tilde{\mathscr{D}}_n(x, y, z)|\, dz \leqslant C \frac{1}{\sqrt{(1-x^2)(1-y^2)}} \mathscr{N}_n \ln \frac{n+1}{\mathscr{N}_n}$$
$$\tilde{L}_n(x) = \int_{-1}^{1} \int_{-1}^{1} |\tilde{\mathscr{D}}_n(x, y, z)|\, dy\, dz \leqslant C \frac{1}{\sqrt{1-x^2}} \mathscr{N}_n \ln \frac{n+1}{\mathscr{N}_n}. \tag{3.5}$$

The constants $C>0$ in the relations (3.5) are independent of $n \in \mathbb{Z}_+$ and $x, y \in (-1, 1)$. In fact, the first estimation (3.5) can be deduced from (3.1), and, consequently,

$$\int_{-1}^{1} \int_{-1}^{1} |\tilde{\mathscr{D}}_n(x, y, z)|\, dy\, dz \leqslant C \int_{-1}^{1} \frac{1}{\sqrt{(1-x^2)(1-y^2)}} \mathscr{N}_n \ln \frac{n+1}{\mathscr{N}_n}\, dy$$
$$\leqslant C \frac{1}{\sqrt{1-x^2}} \mathscr{N}_n \ln \frac{n+1}{\mathscr{N}_n}.$$

This shows the validity of Corollary 3.2.

We consider the pointwise estimation for the partial sums of the following Fourier expansions

$$\sum_{k=0}^{\infty} \hat{f}_k\, p_k(x)\, p_k(y), \qquad \hat{f}_k = \int_{-1}^{1} f(z)\, p_k(z)\, w(z)\, dz \ (k \in \mathbb{Z}_+)$$

and

$$\sum_{k=0}^{\infty} \hat{f}_{kk}\, p_k(x),$$

$$\hat{f}_{kk} = \int_{-1}^{1} \int_{-1}^{1} f(y, z)\, p_k(y)\, p_k(z)\, w(y)\, w(z)\, dy\, dz \ (k \in \mathbb{Z}_+).$$

This problem arises, for example, in the Fourier method for partial difference equations [23, pp. 121–124].

COROLLARY 3.3. *Let* ONSP $\{p_n\}$ $(n \in \mathbb{Z}_+)$ *satisfy* (1.3) *and* (2.9). *Then the following statements are valid*:

(1) *at every* $x, y \in (-1, 1)$ *the following estimation*

$$\left| \sum_{k=0}^{n} \hat{f}_k\, p_k(x)\, p_k(y) \right|$$
$$\leqslant C\, \|f\varphi w\|_\infty \frac{\varphi(x)}{\sqrt{1-x^2}} \frac{\varphi(y)}{\sqrt{1-y^2}} \mathcal{N}_n \ln \frac{n+1}{\mathcal{N}_n} \qquad (n \in \mathbb{Z}_+)$$

is valid, where the constant $C > 0$ *is independent of* f, $n \in \mathbb{Z}_+$, *and* $x, y \in (-1, 1)$;

(2) *at every* $x \in (-1, 1)$ *the estimation*

$$\left| \sum_{k=0}^{n} \hat{f}_{kk}\, p_k(x) \right| \leqslant C \frac{\varphi(x)}{\sqrt{1-x^2}} \mathcal{N}_n \ln \frac{n+1}{\mathcal{N}_n}$$
$$\operatorname*{Sup}_{y, z \in (-1,1)} [|f(y, z)|\, \varphi(y)\, \varphi(z)\, w(y)\, w(z)]$$

holds; *the constant* $C > 0$ *here is independent of the function* f *and the variables* $n \in \mathbb{Z}_+$, $x \in (-1, 1)$.

It can be seen without difficulty that

$$\sum_{k=0}^{n} \hat{f}_k\, p_k(x)\, p_k(y) = \int_{-1}^{1} f(z)\, \mathcal{D}_n(x, y, z)\, w(z)\, dz$$

and

$$\sum_{k=0}^{n} \hat{f}_{kk}\, p_k(x) = \int_{-1}^{1} \int_{-1}^{1} f(y, z)\, \mathscr{D}_n(x, y, z)\, w(y)\, w(z)\, dy\, dz,$$

from which by (3.5) *the results follow.*

Remark. The methods of Section 3 give us an opportunity to investigate ONSP $\{p_n\}$ $(n \in \mathbb{Z}_+)$ for which, instead of (2.9), the estimation

$$|p_n(x)| \leqslant M_n \varphi(x) \qquad (n \in \mathbb{Z}_+;\ -1 < x < 1)$$

holds, but the right-hand side of (3.5) becomes more complicated.

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